Published by Institute of Physics Publishing for SISSA

Received: *November 13, 2007*
Accepted: *December 4, 2007*
Published: *December 12, 2007*

Dressing the giant gluon

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Abstract: We demonstrate the applicability of the dressing method to the problem of constructing new classical solutions for Euclidean worldsheets in anti-de Sitter space. The motivation stems from recent work of Alday and Maldacena, who studied gluon scattering amplitudes at strong coupling using a generalization of a particular worldsheet found by Kruczenski whose edge traces a path composed of light-like segments on the boundary of AdS. We dress this 'giant gluon' to find new solutions in AdS_3 and AdS_5 whose edges trace out more complicated, timelike curves on the boundary. These solutions may be used to calculate certain Wilson loops via AdS/CFT.

Keywords: Integrable Equations in Physics, Bosonic Strings, AdS-CFT Correspondence.

JHEP12(2007)047

Contents

1. Introduction

Classical string solutions play an important role in exploring the AdS/CFT correspondence (see [1] and [2 − 4] for reviews). Generally speaking such solutions fall into two categories. On the one hand there are closed string energy eigenstates in AdS, which are in correspondence with gauge invariant operators of definite scaling dimension in the dual gauge theory. On the other hand we can also consider open strings which end along some curve on the boundary of AdS, corresponding to Wilson loops [5, 6].

An important example of the former is the so-called 'giant magnon' of Hofman and Maldacena [7], which is dual to a single elementary excitation in the gauge theory picture. More general states containing arbitrary numbers of bound or scattering states of magnons correspond to more general classical string solutions [8 − 10]. These solutions can be constructed algebraically using the dressing method [10, 11], a well-known technique [12, 13] for generating solutions of classically integrable equations.

In this paper we turn our attention to the latter, demonstrating the applicability of the dressing method to the problem of constructing certain new Euclidean minimal area surfaces in anti-de Sitter space.[1] To apply the dressing method it is necessary to choose some solution of the classical equations of motion to use as the 'vacuum', which is then 'dressed' to build more general solutions. For the giant magnon system considered in [10, 11] it was natural to choose as vacuum the solution describing a pointlike string moving at

[1]The dressing method has also been used to construct Minkowskian worldsheets in de Sitter space [14, 15].

JHEP12(2007)047

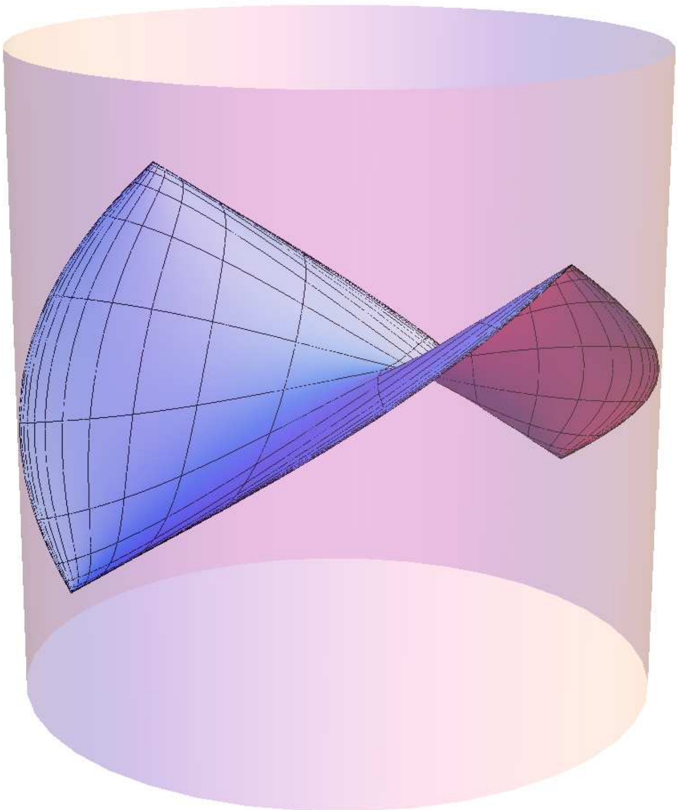

Figure 1: The 'giant gluon' solution (3.1) in AdS_3 global coordinates. The gluons follow the four light-like segments on the boundary of AdS_3 where the worldsheet ends.

the speed of light around the equator of the S^5, since this state corresponds to the natural vacuum in the spin chain picture.

For the present problem we choose as vacuum a particular solution, shown in figure 1, originally used by Kruczenski [16] to study the cusp anomalous dimension via AdS/CFT. It is the minimal area surface which meets the boundary of global AdS_3 along four intersecting light-like lines. This solution was recently generalized, and given a new interpretation, by Alday and Maldacena [17], who gave a prescription for computing planar gluon scattering amplitudes in $\mathcal{N} = 4$ Yang-Mills at strong coupling using the AdS/CFT correspondence and found perfect agreement with the structure predicted on the basis of previously conjectured iteration relations for perturbative multiloop gluon amplitudes [18 – 22].

The Alday-Maldacena prescription is (classically) computationally equivalent to the problem of evaluating a Wilson loop composed of light-like segments. According to the AdS/CFT dictionary, such a Wilson loop is computed by evaluating the area of the surface in figure 1. The interpretation of this surface in terms of a gluon scattering process suggests calling this kind of solution a 'giant gluon.'

We dress the giant gluon to find new minimal area surfaces in AdS_3 and AdS_5 whose edges trace out more complicated, timelike curves on the boundary of AdS. It is not clear whether these new solutions have any interpretation as a scattering process of the type studied in [17], although they do have straightforward interpretations in terms of Wilson loops. However, when calculating a Wilson loop one usually first specifies a curve on the boundary of AdS and then finds the minimal area surface bounding that curve. In contrast, the dressing method provides the minimal area surface without telling us the curve that it spans, i.e. without telling us which Wilson loop it is calculating. That information must be read off directly by analyzing the solution to see where it reaches the boundary of AdS, a procedure that we will see is rather nontrivial.

The outline of this paper is as follows. In section 2 we demonstrate the applicability of the dressing method, focusing on the AdS_3 case which is simpler because there the problem can be mapped into the SU(1,1) principal chiral model. In section 3 we discuss the dressed giant gluon in AdS_3, display explicit formulas for a special case of the solution, and analyze in detail the edge of the worldsheet on the boundary of AdS_3. In section 4 we turn to the more complicated construction for AdS_5 solutions using the SU(2,2)/SO(4,1) coset model, and present some examples.

The main goal of this paper is to demonstrate the applicability of the dressing method. Although we consider a few examples, they amount to only a small subset of the simplest possible solutions. It would be very interesting to more fully explore the parameter space of solutions that can be obtained. It would also be interesting to evaluate the (regulated) areas of these solutions, thereby calculating the corresponding Wilson loops in gauge theory. The giant gluon shown in figure 1 can actually be related [23], by analytic continuation and a conformal transformation, to a closed string energy eigenstate (a limit of the GKP spinning string [1]). It would be interesting to see whether it is possible to relate more general Euclidean worldsheets of the type we consider to various closed string states.

2. AdS dressing method

The dressing method [12] is a general technique for constructing solutions of classically integrable equations. As we review shortly, at the heart of the method lies the ability to transform nonlinear equations of motion into a linear system for an auxiliary field. Here we apply this very general method to the specific problem of constructing minimal area Euclidean worldsheets in anti de-Sitter space. Initially we restrict our attention to AdS_3, where the problem relates to the SU(1,1) principal chiral model, deferring the slightly more complicated AdS_5 case to section 4. Many of the equations in this section are similar to those appearing in [10, 11], which the reader may consult for further details. The two most significant differences compared to the SU(2) principal chiral model considered in [10] are that we use complex coordinates $z, \bar{z}$ on the worldsheet, which is now Euclidean, and that the indefinite SU(1,1) metric significantly changes the behavior of the solutions compared to SU(2).

We parameterize AdS_d with $d+1$ embedding coordinates $\vec{Y}$ subject to the constraint

$$\vec{Y} \cdot \vec{Y} \equiv -Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + \cdots + Y_{d-1}^2 = -1. \tag{2.1}$$

Minimal area worldsheets are given by solutions to the conformal gauge equations of motion

$$\partial\bar{\partial}\vec{Y} - \vec{Y}\left(\partial\vec{Y}\cdot\bar{\partial}\vec{Y}\right) = 0 \tag{2.2}$$

subject to the Virasoro constraints

$$\partial\vec{Y}\cdot\partial\vec{Y} = \bar{\partial}\vec{Y}\cdot\bar{\partial}\vec{Y} = 0. \tag{2.3}$$

Here and throughout the paper we use complex coordinates

$$z = \frac{1}{2}(u_1 + iu_2), \qquad \bar{z} = \frac{1}{2}(u_1 - iu_2), \tag{2.4}$$

with

$$\partial = \partial_1 - i\partial_2, \qquad \bar{\partial} = \partial_1 + i\partial_2. \tag{2.5}$$

Our first step is to recast the system (2.2), (2.3) into the form of a principal chiral model for a matrix-valued field g satisfying the equation of motion

$$\bar{\partial}A + \partial\bar{A} = 0 \tag{2.6}$$

in terms of the currents

$$A = i\partial g\, g^{-1}, \qquad \bar{A} = i\bar{\partial} g\, g^{-1}. \tag{2.7}$$

Note that the relation

$$\bar{\partial}A - \partial\bar{A} - i[A,\bar{A}] = 0 \tag{2.8}$$

follows automatically from (2.7).

To see how this is done let us consider for simplicity first the AdS_3 case. Here we use the coordinates $\vec{Y}$ to parameterize an element g of SU(1,1) according to

$$g = \begin{pmatrix} Z_1 & Z_2 \\ \bar{Z}_2 & \bar{Z}_1 \end{pmatrix}, \qquad Z_1 = Y_{-1} + iY_0, \qquad Z_2 = Y_1 + iY_2, \tag{2.9}$$

which satisfies

$$g^\dagger M g = M, \qquad M = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.10}$$

and

$$\det g = -\vec{Y}\cdot\vec{Y} = +1. \tag{2.11}$$

It is easy to check that the systems (2.2), (2.3) and (2.6), (2.8) are equivalent to each other under this change of variables.

Next we transform the nonlinear second-order system (2.6), (2.7) for $g(z,\bar{z})$ into a linear, first-order system for an auxiliary field $\Psi(z,\bar{z},\lambda)$ at the expense of introducing a new complex parameter λ called the spectral parameter. Specifically, the two equations (2.6), (2.7) are equivalent to

$$i\partial\Psi = \frac{A\Psi}{1+i\lambda}, \qquad i\bar{\partial}\Psi = \frac{\bar{A}\Psi}{1-i\lambda}. \tag{2.12}$$

For later convenience we have rescaled our definition of λ in this equation by a factor of i compared to the conventions of [10, 11].

To apply the dressing method we begin with any known solution g (which we refer to as the 'vacuum' for the dressing method, though we emphasize that any solution may be chosen as the vacuum) and then solve the linear system (2.12) to find $\Psi(\lambda)$ subject to the initial condition

$$\Psi(\lambda = 0) = g. \tag{2.13}$$

In addition we impose on $\Psi(\lambda)$ the SU(1, 1) conditions

$$\Psi^\dagger(\bar{\lambda}) M \Psi(\lambda) = M, \qquad \det \Psi(\lambda) = 1. \tag{2.14}$$

The purpose of the factor of i mentioned below (2.12) is to avoid the need to take $-\bar{\lambda}$ instead of $\bar{\lambda}$ in the first relation here.

Then we make a 'gauge transformation' of the form

$$\Psi'(\lambda) = \chi(\lambda)\Psi(\lambda). \tag{2.15}$$

If $\chi(\lambda)$ were independent of z and $\bar{z}$ this would be an uninteresting SU(1,1) gauge transformation. Instead we want $\chi(\lambda)$ to depend on z and $\bar{z}$ but in such a way that $\Psi'(\lambda)$ continues to satisfy (2.12) and hence $\Psi'(0)$ provides a new solution to (2.6), (2.8). For AdS_3 it is not hard to show that this is accomplished by taking $\chi(\lambda)$ to have the form

$$\chi(\lambda) = 1 + \frac{\lambda_1 - \bar{\lambda}_1}{\lambda - \lambda_1} P \tag{2.16}$$

where λ_1 is an arbitrary complex parameter and P is a projection operator onto any vector of the form $v_1 \equiv \Psi(\bar{\lambda}_1)v$ for any constant vector v. Concretely, P is therefore given by

$$P = \frac{v_1 v_1^\dagger M}{v_1^\dagger M v_1}. \tag{2.17}$$

As in [10] there is a minor remaining detail that (2.16) has

$$\det \chi(\lambda) = \bar{\lambda}_1/\lambda_1 \tag{2.18}$$

so in order for g' to lie in SU(1,1) rather than U(1,1) we should rescale g' by the constant phase factor $\sqrt{\lambda_1/\bar{\lambda}_1}$ to ensure that it has unit determinant. To summarize, the desired dressed solution is given by

$$g' = \sqrt{\frac{\lambda_1}{\bar{\lambda}_1}} \left[1 + \frac{\lambda_1 - \bar{\lambda}_1}{-\lambda_1} P\right] \Psi(0). \tag{2.19}$$

The real embedding coordinates $\vec{Y}'$ of the dressed solution may then be read off from g' using the parameterization (2.9). The resulting solution is characterized by the complex parameter λ_1 and the choice of the constant vector v.

JHEP12(2007)047

3. AdS_3 solutions

In this section we obtain new solutions for worldsheets in AdS_3 via the dressing method, taking as 'vacuum' the giant gluon solution [16, 17]

$$\vec{Y} = \begin{pmatrix} Y_{-1} \\ Y_0 \\ Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \cosh u_1 \cosh u_2 \\ \sinh u_1 \sinh u_2 \\ \sinh u_1 \cosh u_2 \\ \cosh u_1 \sinh u_2 \end{pmatrix} . \tag{3.1}$$

Using the AdS_3 parameterization (2.9) we find from (2.7) that

$$\begin{aligned} A &= 2 \begin{pmatrix} -\cosh u_2 \sinh u_2 & i \cosh^2 u_2 \\ i \sinh^2 u_2 & +\cosh u_2 \sinh u_2 \end{pmatrix} , \\ \bar{A} &= 2 \left(-\cosh u_2 \sinh u_2 \quad i \sinh^2 u_2 i \cosh^2 u_2 + \cosh u_2 \sinh u_2 \right) . \end{aligned} \tag{3.2}$$

Then a solution to the linear system (2.12) for $\Psi(\lambda)$ is[2]

$$\Psi(\lambda) = \begin{pmatrix} m_- \operatorname{ch} Z \operatorname{ch} u_2 + i m_+ \operatorname{sh} Z \operatorname{sh} u_2 & m_- \operatorname{sh} Z \operatorname{ch} u_2 + i m_+ \operatorname{ch} Z \operatorname{sh} u_2 \\ m_+ \operatorname{sh} Z \operatorname{ch} u_2 - i m_- \operatorname{ch} Z \operatorname{sh} u_2 & m_+ \operatorname{ch} Z \operatorname{ch} u_2 - i m_- \operatorname{sh} Z \operatorname{sh} u_2 \end{pmatrix} \tag{3.3}$$

where

$$m_+ = 1/m_- = \left(\frac{1+i\lambda}{1-i\lambda} \right)^{1/4} , \qquad Z = m_-^2 z + m_+^2 \bar{z} . \tag{3.4}$$

The solution (3.3) has been chosen to satisfy the desired constraints (2.14) as well as the initial condition

$$\Psi(0) = \begin{pmatrix} \cosh u_1 \cosh u_2 + i \sinh u_1 \sinh u_2 & \sinh u_1 \cosh u_2 + i \cosh u_1 \sinh u_2 \\ \sinh u_1 \cosh u_2 - i \cosh u_1 \sinh u_2 & \cosh u_1 \cosh u_2 - i \sinh u_1 \sinh u_2 \end{pmatrix} , \tag{3.5}$$

correctly reproducing the giant gluon solution (3.1) embedded into SU(1,1) according to (2.9). The dressed solution g' is then given by (2.19).

3.1 A special case

Since the general solution is rather complicated, we present here an explicit formula for the dressed solution for the particular choice of initial vector $v = (\,1 \quad i\,)$, with λ_1 arbitrary. We find that the dressed SU(1,1) principal chiral field takes the form

$$g' = \begin{pmatrix} Z_1' & Z_2' \\ \bar{Z}_2' & \bar{Z}_1' \end{pmatrix} \tag{3.6}$$

where

$$Z_1' = \frac{1}{|\lambda_1|} \frac{\vec{Y} \cdot \vec{N}_1}{D} , \qquad Z_2' = \frac{1}{|\lambda_1|} \frac{\vec{Y} \cdot \vec{N}_2}{D} \tag{3.7}$$

[2]We will occasionally use sh, ch instead of sinh, cosh to compactify otherwise lengthy formulas.

JHEP12(2007)047

in terms of the numerator factors

$$\vec{N}_1 = \begin{pmatrix} -(\bar\lambda_1|m|^2-\lambda_1)\cosh(Z+\bar Z)+i(\bar\lambda_1|m|^2+\lambda_1)\sinh(Z-\bar Z) \\ -(\lambda_1|m|^2+\bar\lambda_1)\sinh(Z-\bar Z)-i(\lambda_1|m|^2-\bar\lambda_1)\cosh(Z+\bar Z) \\ (\lambda_1-\bar\lambda_1)\bar m(\sinh(Z+\bar Z)-i\cosh(Z-\bar Z)) \\ (\lambda_1-\bar\lambda_1)m(\cosh(Z-\bar Z)-i\sinh(Z+\bar Z)) \end{pmatrix},$$

$$\vec{N}_2 = \begin{pmatrix} -(\lambda_1-\bar\lambda_1)\bar m(\sinh(Z+\bar Z)-i\cosh(Z-\bar Z)) \\ -(\lambda_1-\bar\lambda_1)m(\cosh(Z-\bar Z)-i\sinh(Z+\bar Z)) \\ +(\bar\lambda_1|m|^2-\lambda_1)\cosh(Z+\bar Z)-i(\bar\lambda_1|m|^2+\lambda_1)\sinh(Z-\bar Z) \\ +(\lambda_1|m|^2+\bar\lambda_1)\sinh(Z-\bar Z)+i(\lambda_1|m|^2-\bar\lambda_1)\cosh(Z+\bar Z) \end{pmatrix}, \tag{3.8}$$

$\vec{Y}$ given in (3.1), and the denominator

$$D = (|m|^2-1)\cosh(Z+\bar Z)-i(|m|^2+1)\sinh(Z-\bar Z). \tag{3.9}$$

In these expressions

$$m = \left(\frac{1+i\lambda_1}{1-i\lambda_1}\right)^{1/2}, \qquad \bar m = \left(\frac{1-i\bar\lambda_1}{1+i\bar\lambda_1}\right)^{1/2}, \tag{3.10}$$

and

$$Z = z/m + m\bar z, \qquad \bar Z = \bar z/\bar m + \bar m z. \tag{3.11}$$

The real embedding coordinates $\vec{Y}'$ of the dressed solution are easily read off from (3.6) using (2.9). In figure 2 we plot a representative example of the solution (3.7). However before one can make sense of the plot we must understand the behavior of (3.7) at the boundary of AdS, which we address in the next subsection.

3.2 In search of the Wilson loop

Minimal area worldsheets in AdS_5 are related to Wilson loops in the dual gauge theory [5, 6]. According to the AdS/CFT dictionary, in order to calculate the expectation value of the Wilson loop for some closed path $\mathcal{C}$ on the boundary of AdS we should first find the minimal area surface (or surfaces) in AdS which spans that curve and then calculate e^{-A} where A is the (regulated) area of the minimal surface.

The solutions we have obtained by the dressing method turn this procedure on its head. In the previous subsection we displayed an explicit example of such a solution, which indeed describes a minimal area Euclidean worldsheet in AdS_3, but it is not immediately clear what the corresponding curve $\mathcal{C}$ is whose Wilson loop the solution computes. In order to answer this question we must look at (3.7) and find the locus $\mathcal{C}$ where the worldsheet reaches the boundary of AdS_3—this will tell us which Wilson loop we are computing.

In global AdS coordinates, the familiar radial coordinate ρ is related to the coordinates appearing in (2.9) according to

$$\cosh^2\rho = |Z_1|^2, \qquad \sinh^2\rho = |Z_2|^2. \tag{3.12}$$

Hence the boundary of AdS_3 lies at $Z_i = \infty$. Before proceeding with our complicated dressed solution let us pause to note that the giant gluon solution (3.1) reaches the boundary

JHEP12(2007)047

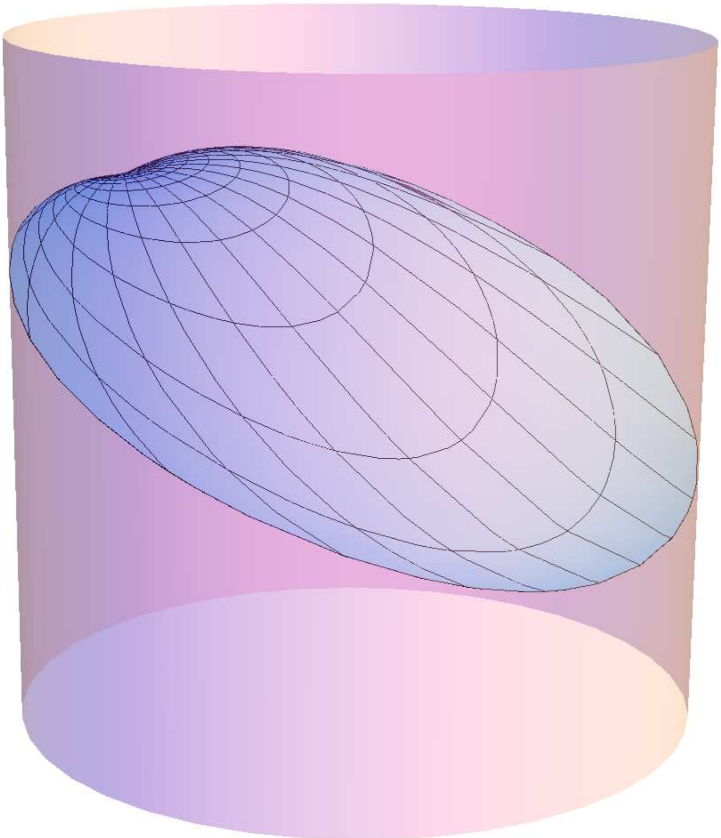

Figure 2: An example of a surface described by the solution (3.7) for the particular choice $\lambda_1 = 1/2 + i/3$.

of AdS_3 precisely when $|u_1| \to \infty$ or $|u_2| \to \infty$. Moreover the four 'edges' of the worldsheet, at $u_1 \to +\infty$, $u_1 \to -\infty$, $u_2 \to +\infty$ and $u_2 \to -\infty$, sit on four separate null lines on the boundary of AdS_3 which intersect each other at four cusps [16, 17] to form the closed curve $\mathcal{C}$.

Looking at the dressed solution (3.7) we see a feature which makes it significantly more complicated to understand than the giant gluon. The presence of the nontrivial denominator factor

$$D = (|m|^2 - 1)\cosh(Z + \bar{Z}) - i(|m|^2 + 1)\sinh(Z - \bar{Z}) \tag{3.13}$$

in (3.7) means that the solution reaches the boundary of AdS_3 any time $D = 0$, which occurs at finite (rather than infinite) values of the worldsheet coordinates $z, \bar{z}$. In fact since D is periodic in Z (with period πi), the solution reaches the boundary of AdS_3 infinitely many times as we allow z (and hence Z) to vary across the complex plane. It is important to note that while D is periodic, the full solution is not.

If we define real variables U_i according to

$$Z = (U_1 + iU_2)/2, \qquad \bar{Z} = (U_1 - iU_2)/2 \tag{3.14}$$

then the locus $\tilde{C}$ of points on the worldsheet where the solution reaches the boundary of AdS_3 is

$$D = (|m|^2 - 1)\cosh U_1 + (|m|^2 + 1)\sin U_2 = 0. \tag{3.15}$$

This equation describes an infinite array of oval-shaped curves $\tilde{\mathcal{C}}_j$ periodically distributed along the U_2 axis and centered at $(U_1, U_2) = (0, 2\pi j + \pi/2)$. Note that the curves $\tilde{\mathcal{C}}_j$ in the worldsheet coordinates are not to be confused with their images $\mathcal{C}_j$ on the boundary of AdS_3 under the map (3.7). In particular the $\tilde{\mathcal{C}}_j$ are unphysical artifacts of the particular coordinate system we happen to be using on the worldsheet — only the curves $\mathcal{C}_j$ on the boundary are physically meaningful.

To summarize, we find that the solution (3.7) actually describes not one but infinitely many different minimal area surfaces in AdS_3, each spanning a different curve $\mathcal{C}_j$ on the boundary. In order to isolate any given worldsheet j we restrict the worldsheet coordinates U_1, U_2 to range over the interior of the curve $\tilde{\mathcal{C}}_j$. In particular, in order to find the area of the j-th worldsheet, and hence calculate the expectation value of the Wilson loop corresponding to the curve $\mathcal{C}_j$, one should integrate the induced volume element on the worldsheet only over the region $\tilde{\mathcal{C}}_j$. It would be interesting to pursue this calculation further, although we will not do so here.

3.3 A very special case

In the previous subsection we explained that the minimal area surfaces generated by the dressing method actually calculate infinitely many different Wilson loops. In general the solutions are sufficiently complicated that we find it necessary to analyze them numerically (one example is shown in figure 3), but it is satisfying to analyze in detail one particularly simple example based on the solution (3.7) which itself is already a special case of the most general dressed solution.

Therefore we look now at the case $\lambda_1 = i$. Since the solution naively looks singular at this value we will carefully take the limit as $\lambda_1 \to i$ from inside the unit circle. To this end we consider

$$\lambda_1 = ia, \qquad m = \sqrt{\frac{1-a}{1+a}} \tag{3.16}$$

in the limit $a \to 1$. In this limit the equation for the boundary reduces to

$$\cosh U_1 = \sin U_2 \tag{3.17}$$

whose solutions are just points in the (U_1, U_2) plane.

In order to isolate what is going on near the point $(0, \pi/2)$ (for example) we should rescale the worldsheet coordinates by defining new coordinates x, y according to

$$U_1 = 2mx, \qquad U_2 = \frac{\pi}{2} + 2my \tag{3.18}$$

Then in the limit $a \to 1$ the equation becomes

$$0 = D = (1 - x^2 - y^2)(1 - a) + \mathcal{O}(1 - a)^2 \tag{3.19}$$

So now the edge of the worldsheet is the circle $x^2 + y^2 = 1$ in the (x, y) plane. Using (3.14) and (3.11) gives

$$u_1 = \frac{1}{2}x(1-a), \qquad u_2 = \frac{\pi}{4a}\sqrt{1-a^2} + \frac{1}{2a}(1-a)y. \tag{3.20}$$

Plugging these values and (3.16) into the solution (3.7) we can then safely take $a \to 1$, obtaining the surface

$$Z_1 = -i\frac{1+x^2+y^2}{1-x^2-y^2}, \qquad Z_2 = \frac{2ix - 2y}{1-x^2-y^2}. \tag{3.21}$$

Switching now to Poincaré coordinates (R, T, X) according to the usual embedding

$$Z_1 = \frac{1}{2}\left(\frac{1}{R} + \frac{R^2 - T^2 + X^2}{R}\right) + i\frac{T}{R}, \qquad Z_2 = \frac{X}{R} + \frac{i}{2}\left(\frac{1}{R} - \frac{R^2 - T^2 + X^2}{R}\right) \tag{3.22}$$

we find

$$R = \frac{1-x^2-y^2}{x}, \qquad T = -\frac{1+x^2+y^2}{x}, \qquad X = -\frac{y}{x}. \tag{3.23}$$

Finally we note that this surface in AdS_3 satisfies

$$-T^2 + X^2 = -1 - \frac{(1-x^2-y^2)^2}{4x^2}. \tag{3.24}$$

At the edge of the worldsheet the second term on the right-hand side is zero, so we conclude that the solution (3.23) intersects the boundary of AdS_3 along the curve described by

$$-T^2 + X^2 = -1. \tag{3.25}$$

Interestingly this is a timelike curve whereas the giant gluon solution we started with traces out a path of lightlike curves on the boundary.

More complicated cases must be studied numerically. In figure 3 we show the timelike curve on the boundary of AdS_3 that bounds the sample surface shown in figure 2.

4. AdS_5 solutions

We now turn our attention to the dressing problem for worldsheets in AdS_5. This case is somewhat more complicated because it is not realized as a principal chiral model. Rather we use the SU(2,2)/SO(4,1) coset model, parameterizing an element g of the coset in terms of the embedding coordinates $\vec{Y}$ according to [24]

$$g = \begin{pmatrix} 0 & +Z_1 & -Z_3 & +\bar{Z}_2 \\ -Z_1 & 0 & +Z_2 & +\bar{Z}_3 \\ +Z_3 & -Z_2 & 0 & -\bar{Z}_1 \\ -\bar{Z}_2 & -\bar{Z}_3 & +\bar{Z}_1 & 0 \end{pmatrix} \tag{4.1}$$

where

$$Z_1 = Y_{-1} + iY_0, \qquad Z_2 = Y_1 + iY_2, \qquad Z_3 = Y_3 + iY_4. \tag{4.2}$$

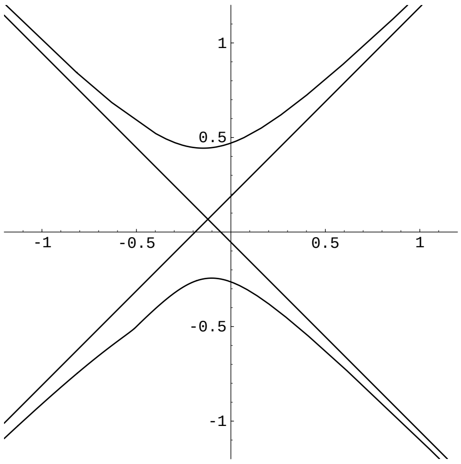

Figure 3: In this plot we consider, as an example, the solution (3.7) for the particular case $\lambda_1 = 1/2 + i/3$. As explained in the text, the solution actually corresponds to infinitely many Wilson loops on the boundary of AdS_3, one of which is the curve shown here in the (X, T) plane on the boundary of AdS_3 in Poincaré coordinates. The light-cone to which these timelike curves asymptote is also shown. The Wilson loop is of course a closed curve; the upper and lower branches shown here live on opposite sides of the AdS_3 cylinder in global coordinates. The minimal area surface spanning this curve is shown in figure 2.

This parameterization satisfies

$$g^{\mathrm{T}} = -g, \qquad g^{\dagger} M g = M, \tag{4.3}$$

where

$$M = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} \tag{4.4}$$

and has determinant

$$\det g = -\vec{Y} \cdot \vec{Y} = 1. \tag{4.5}$$

Taking again the giant gluon solution (3.1) (supplemented with $Y_3 = Y_4 = 0$) as the 'vacuum' we now find that the solution to the linear system (2.12) is

$$\Psi(\lambda) = \begin{pmatrix} 0 & +\operatorname{ch} u_1 \operatorname{ch} U_2 + i m_- \operatorname{sh} u_1 \operatorname{sh} U_2 & 0 & +m_- \operatorname{sh} u_1 \operatorname{ch} U_2 - i \operatorname{ch} u_1 \operatorname{sh} U_2 \\ -\operatorname{ch} U_1 \operatorname{ch} u_2 - i m_+ \operatorname{sh} U_1 \operatorname{sh} u_2 & 0 & +\operatorname{sh} U_1 \operatorname{ch} u_2 + i m_+ \operatorname{ch} U_1 \operatorname{sh} u_2 & 0 \\ 0 & -\operatorname{sh} u_1 \operatorname{ch} U_2 - i m_- \operatorname{ch} u_1 \operatorname{sh} U_2 & 0 & -m_- \operatorname{ch} u_1 \operatorname{ch} U_2 + i \operatorname{sh} u_1 \operatorname{sh} U_2 \\ -m_+ \operatorname{sh} U_1 \operatorname{ch} u_2 + i \operatorname{ch} U_1 \operatorname{sh} u_2 & 0 & +m_+ \operatorname{ch} U_1 \operatorname{ch} u_2 - i \operatorname{sh} U_1 \operatorname{sh} u_2 & 0 \end{pmatrix} \tag{4.6}$$

in terms of

$$U_1 = m_- z + m_+ \bar{z}, \qquad U_2 = (m_- z - m_+ \bar{z})/i, \qquad m_+ = 1/m_- = \left(\frac{1 + i\lambda}{1 - i\lambda}\right)^{1/2}. \tag{4.7}$$

The solution (4.6) has been chosen to satisfy the desired constraints

$$\Psi^\dagger(\bar\lambda)M\Psi(\lambda) = M, \qquad \det\Psi(\lambda) = 1 \tag{4.8}$$

as well as the initial condition

$$\Psi(\lambda = 0) = g, \tag{4.9}$$

where g is the giant gluon solution (3.1) written in the embedding (4.1). Note that the symbols U_1, U_2 defined in (4.7) have been chosen because at $\lambda = 0$ they reduce to u_1, u_2.

4.1 Construction of the dressing factor

The dressing factor for this coset model takes the form

$$\chi(\lambda) = 1 + \frac{\lambda_1 - \bar\lambda_1}{\lambda - \lambda_1}P_1 + \frac{1/\lambda_1 - 1/\bar\lambda_1}{\lambda + 1/\bar\lambda_1}P_2. \tag{4.10}$$

In order to satisfy all the constraints on the dressed solution, we choose P_1 and P_2 as follows. First we choose P_1 to be the hermitian (with respect to the metric M) projection operator onto the vector $v_1 = \Psi(\bar\lambda_1)v$, where v is an arbitrary complex constant vector. Specifically, P_1 is then given as in (2.17) by

$$P_1 = \frac{v_1 v_1^\dagger M}{v_1^\dagger M v_1}, \tag{4.11}$$

which satisfies

$$P_1^2 = P_1, \qquad P_1^\dagger = M P_1 M \tag{4.12}$$

as desired. Next we choose

$$P_2 = \Psi(0)P_1^{\mathrm{T}}\Psi(0)^{-1}. \tag{4.13}$$

Because of (4.12) it is easy to check that P_2 also satisfies

$$P_2^2 = P_2, \qquad P_2^\dagger = M P_2 M, \tag{4.14}$$

so P_2 is also a hermitian projection operator; in fact it is easy to check that P_2 projects onto the vector

$$v_2 = \Psi(0)M\overline{v_1} \tag{4.15}$$

and hence can be written as

$$P_2 = \frac{v_2 v_2^\dagger M}{v_2^\dagger M v_2}. \tag{4.16}$$

Now let us explain the choice (4.13). Notice that

$$v_2^\dagger M v_1 = v_1^{\mathrm{T}} M \Psi(0)^\dagger M v_1 = v_1^{\mathrm{T}}\Psi(0)^{-1}v_1 \tag{4.17}$$

where we used $\Psi(0)^\dagger M\Psi(0) = M$. But since $\Psi(0)$ is antisymmetric, this is zero. So v_2 and v_1 are orthogonal, and hence

$$P_1 P_2 = P_2 P_1 = 0. \tag{4.18}$$

Using all of the above relations one can check that (4.10) satisfies the conditions

$$[\chi(\bar{\lambda})]^\dagger M \chi(\lambda) = M, \qquad \Psi^{\mathrm{T}}(0)\chi^{\mathrm{T}}(0) = -\chi(0)\Psi(0), \tag{4.19}$$

which guarantee that the dressed solution $\Psi'(\lambda) = \chi(\lambda)\Psi(\lambda)$ continues to satisfy (4.3). As in the AdS_3 case we find that χ does not have unit determinant but rather

$$\det \chi(\lambda) = \frac{\lambda - \bar{\lambda}_1}{\lambda - \lambda_1}\frac{\lambda - 1/\lambda_1}{\lambda - 1/\bar{\lambda}_1}. \tag{4.20}$$

We must therefore rescale the dressed solution $\Psi'(0) = \chi(0)\Psi(0)$ by a factor of $\sqrt{\lambda_1/\bar{\lambda}_1}$. To summarize, the dressed solution g' is given by

$$g' = \sqrt{\frac{\lambda_1}{\bar{\lambda}_1}}\left[1 + \frac{\lambda_1 - \bar{\lambda}_1}{-\lambda_1}P_1 + \frac{1/\lambda_1 - 1/\bar{\lambda}_1}{1/\bar{\lambda}_1}P_2\right]\Psi(0) \tag{4.21}$$

in terms of (4.6) and the projection operators (4.11), (4.13). The solution is characterized by an arbitrary complex parameter λ_1 and the choice of a complex four-component vector v.

4.2 A special case

Since the general solution is again rather complicated we display only a special case, choosing the vector $v = (\,1 \quad i \quad 0 \quad 0\,)$. We then find that the dressed solution g' has the form (4.1) with

$$Z_1' = \frac{1}{|\lambda_1|}\frac{\vec{Y}\cdot\vec{N}_1}{D}, \qquad Z_2' = \frac{1}{|\lambda_1|}\frac{\vec{Y}\cdot\vec{N}_2}{D}, \qquad Z_3' = \frac{1}{|\lambda_1|}\frac{N_3}{D} \tag{4.22}$$

in terms of the numerator factors

$$\begin{aligned}
\vec{N}_1 &= \begin{pmatrix} -|m|^2\bar{\lambda}_1(\operatorname{ch}U_1\operatorname{ch}\bar{U}_1 + \operatorname{ch}U_2\operatorname{ch}\bar{U}_2) + \lambda_1\operatorname{sh}U_1\operatorname{sh}\bar{U}_1 + |m|^4\lambda_1\operatorname{sh}U_2\operatorname{sh}\bar{U}_2 \\ -i|m|^2\lambda_1(\operatorname{ch}U_1\operatorname{ch}\bar{U}_1 + \operatorname{ch}U_2\operatorname{ch}\bar{U}_2) + i\bar{\lambda}_1\operatorname{sh}U_1\operatorname{sh}\bar{U}_1 + i|m|^4\bar{\lambda}_1\operatorname{sh}U_2\operatorname{sh}\bar{U}_2 \\ -(\lambda_1 - \bar{\lambda}_1)\bar{m}\operatorname{sh}U_1\operatorname{ch}\bar{U}_1 + i(\lambda_1 - \bar{\lambda}_1)\bar{m}|m|^2\operatorname{ch}U_2\operatorname{sh}\bar{U}_2 \\ +i(\lambda_1 - \bar{\lambda}_1)m\operatorname{ch}U_1\operatorname{sh}\bar{U}_1 - (\lambda_1 - \bar{\lambda}_1)m|m|^2\operatorname{sh}U_2\operatorname{ch}\bar{U}_2 \end{pmatrix}, \\
\vec{N}_2 &= \begin{pmatrix} -(\lambda_1 - \bar{\lambda}_1)\bar{m}\operatorname{sh}U_1\operatorname{ch}\bar{U}_1 + i(\lambda_1 - \bar{\lambda}_1)\bar{m}|m|^2\operatorname{ch}U_2\operatorname{sh}\bar{U}_2 \\ +i(\lambda_1 - \bar{\lambda}_1)m\operatorname{ch}U_1\operatorname{sh}\bar{U}_1 - (\lambda_1 - \bar{\lambda}_1)m|m|^2\operatorname{sh}U_2\operatorname{ch}\bar{U}_2 \\ -|m|^2\bar{\lambda}_1(\operatorname{ch}U_1\operatorname{ch}\bar{U}_1 + \operatorname{ch}U_2\operatorname{ch}\bar{U}_2) + \lambda_1\operatorname{sh}U_1\operatorname{sh}\bar{U}_1 + |m|^4\lambda_1\operatorname{sh}U_2\operatorname{sh}\bar{U}_2 \\ -i|m|^2\lambda_1(\operatorname{ch}U_1\operatorname{ch}\bar{U}_1 + \operatorname{ch}U_2\operatorname{ch}\bar{U}_2) + i\bar{\lambda}_1\operatorname{sh}U_1\operatorname{sh}\bar{U}_1 + i|m|^4\bar{\lambda}_1\operatorname{sh}U_2\operatorname{sh}\bar{U}_2 \end{pmatrix}, \\
N_3 &= \bar{m}(\lambda_1 - \bar{\lambda}_1)(-i\operatorname{sh}U_1\operatorname{ch}\bar{U}_2 + |m|^2\operatorname{ch}U_1\operatorname{sh}\bar{U}_2),
\end{aligned} \tag{4.23}$$

$\vec{Y}$ again given in (3.1), and the denominator

$$D = -|m|^2(\operatorname{ch}U_1\operatorname{ch}\bar{U}_1 + \operatorname{ch}U_2\operatorname{ch}\bar{U}_2) + \operatorname{sh}U_1\operatorname{sh}\bar{U}_1 + |m|^4\operatorname{sh}U_2\operatorname{sh}\bar{U}_2. \tag{4.24}$$

In these expressions m and $\bar{m}$ are as in (3.10), with

$$U_1 = z/m + m\bar{z}, \qquad U_2 = (z/m - m\bar{z})/i. \tag{4.25}$$

The real embedding coordinates $\vec{Y}'$ of the dressed solution may then be extracted from (4.2). It is straightforward, though somewhat tedious, to directly verify that the resulting $\vec{Y}'$ satisfies the equations of motion (2.2) and the Virasoro constraints (2.3), providing a check on our application of the dressing method.

JHEP12(2007)047

Acknowledgments

We are grateful to J. Maldacena for comments. The work of AJ is supported in part by the Department of Energy under contract DE-FG02-91ER40688. The work of MS is supported by NSF grant PHY-0610259 and by an OJI award under DOE contract DE-FG02-91ER40688. The work of AV is supported by NSF CAREER Award PHY-0643150.

A. Conventions

Here we summarize the standard conventions for global AdS_3 that we have used in preparing figures 1 and 2. We parametrize the SU(1,1) group element (2.9) as

$$g = \begin{pmatrix} e^{+i\tau} \sec\theta & e^{+i\phi} \tan\theta \\ e^{-i\phi} \tan\theta & e^{-i\tau} \sec\theta \end{pmatrix}, \tag{A.1}$$

where τ is global time, ϕ is the azimuthal angle, and θ runs from 0 in the interior of the AdS_3 cylinder to $\pi/2$ at the boundary of AdS_3. In terms of these quantities the parametric plots in figures 1 and 2 have Cartesian coordinates

$$(x, y, z) = (\theta \cos\phi, \theta \sin\phi, \tau) \tag{A.2}$$

and the boundary of AdS_3 is the cylinder $x^2 + y^2 = (\pi/2)^2$.

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